Matrix Code

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1. Introduction

By *imperative programming* we will understand the writing of code in which the state of the computation is directly tested and explicitly manipulated by assignment statements. As a programming paradigm, imperative programming should be compared with functional and logic programming. Compared to these latter paradigms, imperative programming is in an unsatisfactory state. At least as a first approximation, a definition in functional or logic programming is both a specification and is executable. In imperative programming proving that a function body meets its specification is such a challenge that it is not considered part of a programmer's task. Another difference, probably related, is that functional and logic programming have an elegant mathematical semantics in which the behaviour of an executable definition is characterized as a fixpoint of the transformation associated with the definition.

C is the programming language *par excellence* for imperative programming. But in C one can fake functional programming to a certain extent by doing as much as possible with function definitions, function calls, and function parameters. In this paper we will be concerned with what may be called *hard-core* imperative code: code in the form of the body of a procedure (`void` function in C) that contains no global variables and that interacts only with its environment by reading, testing, and modifying the actual parameters in the call. These parameters, together with any local variables that may be present, comprise the state that is changed by assignment statements. Surprisingly perhaps, hard-core imperative code does not exclude function or procedure calls.

In imperative programming verification is a serious problem. The problem is more serious than in functional or logic programming because there the executable code can itself be the definition of the function or predicate to be executed. Of course this ideal is rarely reached completely. But it is a clear ideal for the programmer to strive after. In imperative programming such an ideal does not exist. Here correctness has to be proved independently of the code. Although a powerful verification method was developed by Floyd and by Hoare, the experience is that it is hard to produce a correctness proof for *existing* code. Dijkstra observed [7, 8] that code has to be *designed* for correctness proof. He did not make this suggestion more concrete than to call for the *parallel development of proof and code*.

This paper is a contribution to the parallel development of proof and code for imperative programming. It takes the form of a new language, called *Matrix Code*, in which programs take the form of a matrix of which the elements are binary relations among data states. Matrix Code is distinguished by a development process that begins with a null code matrix, progresses with small, obvious steps, and ends with a matrix that is of a special form that is trivially translatable to a conventional language like Java or `C`. The result of the translation has the same behaviour as the one determined by the mathematical semantics of the code matrix. Therefore the latter can be said to be executable. As every stage in the development process is partially correct with respect to the specification (the correctness of the initial null code matrix is *very* partial). Matrix Code comes close to the ideal in which the code is itself a proof of partial correctness. Matrix Code comes with an abstract machine, which we call a *dual-state machine* (DSM). The DSM has easily identifiable special cases that are trivially translatable to conventional languages like `C` or Java.

Plan of the paper. Because we derive DSMs from finite-state machines we first review conventional automata theory and regular expressions. The step to DSMs is made by exploiting the fact that formal languages are mathematically similar to binary relations and that both are best regarded as interpretations of regular expressions. Accordingly, in Section 2 we establish our notation and terminology for formal languages, binary relations, and regular expressions.

In Section 3 we present the main definitions concerning FSMs. This presentation is necessary because of slight, yet essential variations in the usual definitions. One such variation is that the transition is given as a matrix. In Section 4 dual-state machines are introduced as a close variant of FSMs. The versatility of DSMs is demonstrated by examples including an FSM, a Turing machine, and DSMs that translate to `C` programs for generating prime numbers and for merging files. As the latter type of DSM is the motivation for the entire enterprise we devote Section 5 to it.

In Section 6 we adapt the verification method of Floyd and Hoare to Matrix Code. In Section 7 we solve as example problem the generation of prime numbers in the systematic manner that is specific to Matrix Code. This is the same problem as one of those treated by Dijkstra in [9], so that Matrix Code can be compared with structured programming. Although the derivation method for the prime-number algorithm is original, the computations of the resulting code are the same as those of the conventionally produced version. But Matrix Code is not only valuable as a method for developing proof and code in parallel, but, as we show in the derivation of the merging algorithm in Section 8, it is valuable also for finding algorithms that are more efficient than those obtained in the conventional manner. The final two sections draw conclusions and survey related work in widely scattered areas of computer science.

2. Preliminaries

The dual-state machines to be introduced in this paper are a variant of the classical finite-state machines. Just as finite-state machines define formal languages, dual-state machines define binary relations. The similarity between the two types of machine has to do with the similarity between formal languages and binary relations. One of the ways this similarity manifests itself is the fact that formal languages and binary relations have a natural notation in common: regular expressions.

2.1. Formal languages

Given a set A, we denote the set of finite sequences of its elements as A^*. We often think of A as an "alphabet", of its elements as "symbols", of the sequences of symbols as "words", and of sets of words as a (formal) "language".

A^* includes the empty word, the sequence of length 0, which is denoted e. The null language is the empty set. This is not to be confused with the unit language, which contains the empty word as its only word.

The concatenation of words w_0 and w_1 is denoted $w_0 \cdot w_1$. We have $e \cdot w = w = w \cdot e$ for all words w. Concatenation of words is extended elementwise to concatenation of languages: $L_0 \cdot L_1 = \{w_0 \cdot w_1 \mid w_0 \in L_0 \land w_1 \in L_1\}$. Concatenation of a language L with itself gives rise to the powers of L: L^0 is the unit language and $L^i = L \cdot L^{i-1}$ for all $i > 0$. The closure * of L is defined as $L^* = \bigcup_{n=0}^{\infty} L^n$.

The partial order $\leq$ on formal languages is defined to be set inclusion among the subsets of A^*.

2.2. Binary relations

A binary relation on a set D is a subset of the Cartesian product $D \times D$. If (d_0, d_1) is in a binary relation, then we say that d_0 is an *input*; d_1 is a corresponding *output* of the relation.

The null relation is the empty subset of $D \times D$. The identity relation I_D on D is $\{(d_0, d_1) \in D \times D \mid d_0 = d_1\}$. The union $R_0 \cup R_1$ of binary relations R_0 and R_1 is defined to be their union as subsets of $D \times D$. The composition $R_0; R_1$ of binary relations R_0 and R_1 is $\{(d_0, d_1) \in D \times D \mid \exists d \in D.\ (d_0, d) \in R_0 \wedge (d, d_1) \in R_1\}$.

Powers of a relation R are defined by $R^n = R; R^{n-1} = R^{n-1}; R$ for $n > 0$ and $R^n = I_D$ for $n = 0$. We write R^* for $\bigcup_{n=0}^{\infty} R^n$.

The partial order $\leq$ on binary relations is defined to be set inclusion among the subsets of $D \times D$.

2.3. Regular expressions

The syntax of regular expressions [16] over a given set of constants is defined as follows.

1. The constants, 0, and 1 are regular expressions.
2. If E and F are regular expressions then so are $E + F$ and $E \cdot F$. nE and E^n are shorthand for

$$\underbrace{E + \ldots + E}_{n \text{ times}} \text{ and } \underbrace{E \cdot \ldots \cdot E}_{n \text{ times}} \text{ for } n > 0.$$

   $0 \cdot E$ is 0 and E^0 is 1.
3. If S is a finite set of regular expressions and E is a regular expression, then ΣS is defined as 0 if $S = \emptyset$ and $\Sigma(S \cup \{E\}) = E + \Sigma S$.
4. If E is a regular expression, then so is its closure E^*.

In practice a different syntax is used for regular expressions. We see `EF` for $E \cdot F$, `E|F` for $E + F$, `E?` for $0 + E$, and `E+` for $E \cdot E^*$.

The syntax of regular expressions has several semantics: algebras of which the elements and operations can serve as interpretations of regular expressions. Here these algebras are formal languages and binary relations and serve as semantics for regular expressions. The way we intend these algebras to be semantics for regular expressions is shown in Figure 1.

The following equalities in terms of regular expressions hold for both formal languages and binary relations as interpretation [5].

$$E + F = F + E,\ E + E = E,\ E \cdot (F + G) = E \cdot F + E \cdot G,$$

regular expressions	0	1	$+$	$\cdot$	$*$
formal languages	empty language	$\{e\}$	$\cup$	$\cdot$	$*$
binary relations	empty subset of $D \times D$	I_D	$\cup$	;	$*$

Figure 1: Formal languages and binary relations as semantics for regular expressions.

$$
\begin{aligned}
&(F+G)\cdot E = F\cdot E + G\cdot E,\ E\cdot(F\cdot G) = (E\cdot F)\cdot G,\\
&0+E = E+0 = E,\ 1\cdot E = E\cdot 1 = E,\ 0\cdot E = E\cdot 0 = 0,\\
&(E+F)^* = (E^*\cdot F)\cdot E^*,\ (E\cdot F)^* = 1 + E\cdot(F\cdot E)^*\cdot E,\\
&(E^*)^* = E^*,\ E^* = (E^n)^*\cdot E^{<n}
\end{aligned}
$$

In the last equality $E^{<n}$ denotes the product of all k such that $0 \leq k < n$. The axiom states that every power E^i can be written in the form $(E^n)^j \cdot E^k$ with $0 \leq k < n$.

In addition the partial order $\leq$ defined in the two interpretations for regular expressions have the property of being monotonic with respect to the operations.

3. Finite-state machines

We find the classical finite-state machine useful as the precursor of the dual-state machine, a related device with many interesting special cases, including small imperative programs, such as are suitable for the bodies of functions in conventional programming languages.

Our starting point is the nondeterministic finite-state machine as a device for recognizing a certain class of languages. The syntax of finite-state machines is as follows.

Definition 1. *A finite-state machine (FSM) is a tuple (K, A, δ, S, H) where K and A are nonempty finite sets, δ is a function of type $K \times K \to W$, where W is the set of finite sets of words over A; $S \in K$, and $H \in K$. The elements of K are called* states*; the elements of A, the alphabet, are called* symbols*; δ is the* transition matrix*, a matrix with rows and columns indexed* *by the elements of K. The transition matrix has sets of words over A as elements. $S \in K$ is the* start state*; $H \in K$ is the* halt state*. For all $k \in K$ it is the case that δ must satisfy $\delta[k, S] = \emptyset$ and $\delta[H, k] = \emptyset$.*

Here we find two departures from the conventional definition: (1) The transition δ is a matrix. In this way we make explicit what is conventionally left implicit. (2) There is a single halt state. As is well-known, this neither adds to nor detracts from the FSM's power. An advantage of the single halt node is that FSM's become composable and that one FSM can be substituted into another.

For an example of an FSM, see Figure 2. We choose the FSM to be nondeterministic as starting point because it is equivalent to some deterministic version and because it is mathematically more tractable.

Definition 1 gives only the syntax of an FSM. A common analogy is to view an FSM as a machine that can do work. This work is to make computations, as given by the semantics below.

Definition 2. *A* configuration *of an FSM (K, A, δ, S, H) is a pair (k, w) where $k \in K$ and $w \in A^*$. A* transition *is a pair $((k_0, w), (k_1, w \cdot u))$ of configurations such that $u \in \delta[k_0, k_1]$. A* computation *of the FSM is a sequence of configurations such that (S, e) is the first element and every pair of successive configurations is a transition of the FSM.*

A computation is complete *if its last configuration is not the first configuration of any transition.*

A complete computation is successful *if its last configuration is (H, w) for some $w \in A^*$; otherwise it is* failed.

$\mathcal{L}(K, A, \delta, S, H)$ is the language accepted by *the FSM with components (K, A, δ, S, H); it is the set of all $w \in A^*$ such there exists a successful computation starting in (S, e).*

In the machine analogy of an FSM the transition $((k_0, w), (k_1, w \cdot u))$ is said to "consume" the word u from the input.

Example. The FSM in Figure 2 has the following as one of its computations

```
(S,e)(A,-)(B,-1)(A,-12)(B,-123)(H,-123)
```

The fact that

```
(S,e)(A,-)(B,-1)(H,-1)
```

is also a computation shows that this FSM is nondeterministic.

□

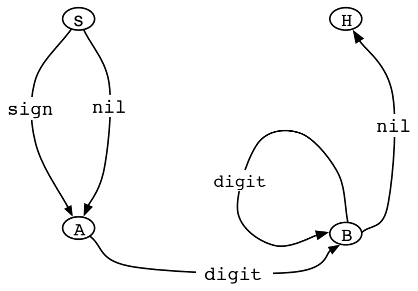
Lemma 1. *An FSM (K, A, δ, S, H) has a computation* `(x,v),...,(y,v.u)` *of length n iff $u \in \delta^n[x,y]$, for all $n = 1, 2, \ldots$*

Proof. Straightforward induction on n.

Theorem 1. $\mathcal{L}(K, A, \delta, S, H) = \delta^*[S, H]$.

Proof. $w \in \mathcal{L}(K, A, \delta, S, H)$ iff there exists a computation $(S, e), \ldots, (H, w)$ iff there exists an n such that $w \in \delta^n[S, H]$ iff $w \in \delta^*[S, H]$.

□



B	A	S	
`nil`			H
		`sign` ∪ `nil`	A
`digit`	`digit`		B

Figure 2: On the left, transition matrix in graph form of an FSM that accepts decimal numerals. There is no arc from state k_0 to state k_1 where $\delta[k_0, k_1] = \emptyset$. The labels on the arcs are `nil` = $\{e\}$, where e is the empty word, `sign` = $\{\langle - \rangle, \langle + \rangle\}$ and `digit` = $\{\langle 0 \rangle, \ldots, \langle 9 \rangle\}$. Here $\langle x \rangle$ is the word of length 1 containing the symbol x. On the right, matrix version of Figure 2. The rows and columns are identified by the states as labels. As the row labeled by `S` is by definition empty, it is omitted. Similarly, the column labeled by `H` is omitted. Empty cells are understood to contain $\emptyset$.

According to Definition 1 a configuration (S, w) can occur only as the first configuration of a computation. Similarly, a configuration (H, w) can occur only as the second of the last configuration of a computation.

The purpose of the computations of an FSM is to define languages in the form of sets of words over the alphabet A.

An FSM is usually presented as a directed graph with the states as nodes and with language $\delta[x, y]$ labeling the arc from state x to state y. We prefer the presentation in Figure 2.

4. Dual-state machines

The operation of an FSM consists of two kinds of changes: a change of state and an advance of the input when a symbol is consumed. Imagine giving the machine a data memory in the form of a file that can be accessed only sequentially. This memory can take the place of the input. The act of consuming a symbol becomes a change in the state of the data in memory. But the elements of K, although not data, are also a kind of memory. To avoid confusion between these different kinds of memory, we rename elements of K to *control states*; the states of data memory can then be called *data states*. A machine with these two kinds of memory we call a *dual-state machine*, a machine that is very like an FSM: note the similarity between Definitions 1 and 3; between Definitions 2 and 4.

The syntax of dual-state machines is defined as follows.

Definition 3. *The syntax of a* dual-state machine *(DSM) is given by a tuple* (K, D, δ, S, H) *where* K *is a nonempty finite set,* D *is a nonempty set,* δ *is a function of type* $K \times K \to 2^{D \times D}$, $S \in K$, *and* $H \in K$. K *consists of* control states*;* D *consists of* data states*;* δ *is the* transition matrix*, a matrix with rows and columns labeled by the elements of* K *with binary relations over* D *as elements.* $S \in K$ *is the* start state*;* $H \in K$ *is the* halt state*. For all* $k \in K$ *it is the case that* δ *must satisfy* $\delta[k, S] = \emptyset$ *and* $\delta[H, k] = \emptyset$.

4.1. The computations of a DSM

The semantics of DSMs is defined as follows.

Definition 4. *A* configuration *of a DSM* (K, D, δ, S, H) *is a pair* (k, d) *where* $k \in K$ *is a control state and* $d \in D$ *is a data state.*

A transition *is a pair* $((k_0, d_0), (k_1, d_1))$ *of configurations such that* $(d_0, d_1) \in \delta[k_0, k_1]$.

A segment *of the DSM is a sequence of configurations such that every pair of successive configurations is a transition of the DSM. The* length *of a segment is the number of transitions in it.*

A computation *of the DSM is a segment in which the first configuration is* (S, d) *for some* $d \in D$[1].

A computation is complete *if its last configuration is not the first configuration of any transition.*

[1]We see that for a configuration (k, d) in a computation to be followed by (k', d') it is necessary that $((k, d), (k', d'))$ is a transition. It is possible that the DSM admits a different transition $((k, d), (k'', d''))$. In other words, DSMs are not necessarily deterministic.

A complete computation is successful *if its last configuration is* (H, d) *for some* $d \in D$*; otherwise it is* failed*.*

$\mathcal{R}(K, D, \delta, S, H)$ *is the* relation computed by *the DSM with components* (K, D, δ, S, H)*; it is the set of all* $(d, d') \in D \times D$ *such there exists a computation* $(S, d), \ldots, (H, d')$*.*

It helps to visualize the Definition 4 in the following way (see Figures 13 and 18). Execution of a code matrix consists of an execution agent performing a sequence of cycles. The agent carries a configuration which is updated during the cycle. At the beginning of the cycle the agent carries the configuration (k, d). It enters the matrix through the column indexed by k until it encounters a non-empty cell. Let r be the index of the row in which this cell occurs and let R be the relation in this cell. If the data state d of the agent is such that there is a $(d, w) \in R$, then the agent exits to the right with configuration (r, w). This completes the cycle, and the agent begins a new cycle unless it exited through row `H`.

The agent may start a cycle in a column that does not contain a transition having its data state as input. In that case the agent does not complete the cycle and execution fails.

Initially the agent carries a configuration with control state `S`. If and when the control state changes to `H`, execution halts with success.

Definition 5. *Given matrices* $M, N \in K \times K \to 2^{D \times D}$ *we define their product* $M; N$ *by* $(M; N)[i, k] = \bigcup_{j \in K} M[i, j]; N[j, k]$ *for all* $i, k \in K$*.*

Let I be the K-labeled matrix of binary relations over D that has the identity relation on D on the main diagonal and the empty relation elsewhere. Then we have $I; M = M; I = M$ with M any K-labeled matrix with binary relations over D as elements. We write M^n for $M^{n-1}; M$ for a positive integer n while $M^0 = I$.

We characterize the relation computed by a DSM in terms of its powers. First a lemma concerning these powers.

Lemma 2. *A DSM with transition matrix* δ *has a computation containing a segment* $(k, d), \ldots, (k', d')$ *of length* n *iff there exists an* n *such that* $(d, d') \in \delta^n[k, k']$*.*

Proof

(If)

By induction on the segment length n. If $n = 1$ the segment has the form

$(k, d), (k', d')$, so that $((k, d), (k', d'))$ is a transition and we have $(d, d') \in \delta[k, k']$ by the definition of computation.

Induction step.

$(d, d'') \in \delta^{n+1}[k, k'']$ implies that there exists an d' and an k' such that $(d, d') \in \delta^n[k, k']$ and $(d', d'') \in \delta[k', k'']$. Hence, by the induction assumption, there exists a computation segment $(k, d), \ldots, (k', d')$ of length n and $(d', d'') \in \delta[k', k'']$, which implies that there exists a segment $(k, d), \ldots, (k'', d'')$ of length $n + 1$.

(Only if)

That $(k_0, d), \ldots, (k_{n-1}, d'), (k_n, d'')$ is a computation implies (by the induction hypothesis) that $(d, d') \in \delta^n[k_0, k_{n-1}]$ and $(d', d'') \in \delta[k_{n-1}, k_n]$. By the definition of relational composition this implies that $(d, d'') \in \delta^n[k_0, k_{n-1}]; \delta[k_{n-1}, k_n]$. We have

$$\delta^n[k_0, k_{n-1}]; \delta[k_{n-1}, k_n] \subseteq \bigcup_{j \in K} \delta^n[k_0, j]; \delta[j, k_n] = (\delta^n; \delta)[k', k''] = \delta^{n+1}[k', k'']$$

This implies that $(d, d'') \in \delta^{n+1}[k, k'']$.

Theorem 2. *$\mathcal{R}(K, D, \delta, S, H) = \delta^*[S, H]$.*

Proof Suppose that the pair (d, d') of data states is in the relation computed by δ. By Definition 4 there exists a computation of δ that begins with (S, d) and ends with (H, d'). According to Lemma 2 there is an n such that $(d, d') \in \delta^n[S, H]$. Hence $(d, d') \in \bigcup_{n=0}^{\infty} \delta^n[S, H]$.

Suppose that $(d, d') \in \bigcup_{n=0}^{\infty} \delta^n[S, H]$. By the finiteness assumptions there exists an n such that $(d, d') \in \delta^n[S, H]$. According to Lemma 2 this implies that there exists a computation of δ that begins with (S, d) and ends with (H, d'). Therefore (d, d') is in the relation computed by δ, according to Definition 4.

4.2. Examples of dual-state machines

Example: FSM as DSM. A dual-state machine can simulate an FSM. When doing so, we must keep in mind that the data state of the DSM is the input of the FSM. This reverses the role of data in the configurations. In FSMs the configuration contains the accepted part of the input, so it grows at transitions. In the DSM simulation the configuration contains the input of the FSM, so it shrinks at transitions.

In this example we get a simulation of the FSM of Figure 2 by setting $K = \{S, A, B, H\}$, D equal to the set of words over$\{+, -, 0, \ldots, 9\}$, and δ equal to the matrix in Figure 2 where `nil` is equal to $\{(w, w) \mid w \in D\}$, `digit` is equal to the set of all $(x \cdot y, y)$ such that x is a word of unit length over the alphabet $\{0, \ldots, 9\}$, and `sign` is equal to the set of all $(x \cdot y, y)$ such that x is a word of unit length over the alphabet $\{-, +\}$.

□

Example: Turing Machine as DSM. We saw that an FSM is a DSM with a certain type of memory defined by the admissible operations. A Turing machine is a DSM with a different type of memory defined by a different set of admissible operations. Among the several variants of Turing machine we choose the one where the memory takes the form of a sequence of squares (a "tape") that is unbounded in both directions. Each square contains one symbol from the finite alphabet A. In addition to the contents of the tape, the state of the memory is determined by a pair (S, D) where S is a square of the tape (the "scanned" square) and D is a direction on the tape, being L (left), R (right), or d (don't care). The operations on the memory include reading, a function with no argument having as value the symbol on the scanned square and writing, a function with a symbol as argument causing the scanned square to contain that symbol. Writing has an additional effect: to "move the tape", meaning that it causes the scanned square to become the one on the left or on the right of the currently scanned square, depending on whether D ("direction") is L or R.

The operation of a Turing machine is determined by a set of rules, each in the form of quintuple $< Q, S, Q', S', D' >$. The rule specifies that, if the state is Q and the scanned square contains S, then S' is written, the tape moves in the current direction D, and the state and direction become Q' and D', respectively.

For this example we selected a simple Turing machine ([14], page 122).

The conventional presentation of the Turing machine as set of quintuples is in Figure 3. The matrix version is in Figure 4. The Matrix Code version is in Figure 7. As a first step for its simulation by a DSM we rewrite the conventional Turing machine presentation to matrix format, which we then find *is* a DSM. Subsequently we rewrite the code matrix to `C` or `C++`.

The Turing machine of Figure 3 is designed to start operation with a tape containing a sequence of parentheses bounded on either side by the symbol A. Initially the scanned square is the square containing the leftmost parenthesis, that is, the square to the right of the leftmost A. When the machine halts, all matching parentheses have been removed. In this way one can tell whether the tape initially contained a well-formed sequence. Thus, for example,

```
A ( ( ( ( ( ( ) ) ) ) A
```

Q	Q_0	Q_0	Q_0	Q_0	Q_1	Q_1	Q_1	Q_1	Q_2	Q_2	Q_2	Q_2
S	)	(	A	X	)	(	A	X	)	(	A	X
Q'	Q_1	Q_0	Q_2	Q_0	Q_1	Q_0	H	Q_1	–	H	H	Q_2
S'	X	(	A	X	)	X	0	X	–	0	1	X
D'	L	R	L	R	L	R	d	L	–	d	d	L

Figure 3: A Turing machine. The leftmost column shows the generic quintuple $< Q, S, Q', S', D' >$. The other twelve columns contain the actual twelve quintuples that define the Turing machine. The states are Q_0, Q_1, Q_2, and H. The tape symbols are), (, A, X, 0, and 1. The dashes indicate that in state Q_2 the symbol ')' is never encountered. The d's in the last row stand for "don't care".

is replaced by

```
A ( 0 X X X X X X X X A,
```

indicating that the input sequence was unbalanced because of an unmatched open parenthesis, whereas

```
A ( ( ( ( ( ) ) ) ( ) ) ) A
```

is replaced by

```
1 X X X X X X X X X X X X A
```

The conventional presentation of Turing machines as a set of quintuples hides their essence, which is a matrix. Just as FSMs centre around transitions from state to state, so do Turing machines. Whatever the nature of this transition, its natural presentation is as an element of a matrix of which the rows and columns are indexed by the states. Figure 2 gives this matrix for an FSM; Figure 4 gives this matrix for the Turing machine in Figure 3.

To familiarize ourselves with the matrix format, let us find in Figure 4 the equivalent of the quintuple $< Q_0,), Q_1, X, L >$ of Figure 3. In this quintuple we see that it specifies a transition from Q_0 to Q_1. The attributes of this transition are in column Q_0, row Q_1. The machine makes this particular transition if the scanned square contains ')'. As a result of the transition an X is written on the scanned square and the tape moves left. Given that the transition is from Q_0 to Q_1, the further particulars can be given in the form of a condition/action rule:

```
) -> X;L,
```

which is in the matrix cell of column Q_0, row Q_1. Some transitions, for example the one from Q_0 to Q_0, contain more than one such rule, one for each of the possible contents of the scanned square. See Figure 4.

□

Q2	Q1	Q0	
(-> 0;d A -> 1;d	A -> 0;d		H
	(-> X;R	(-> (;R X -> X;R	Q0
	) ->);L X -> X;L	) -> X;L	Q1
X -> X;L		A -> A;L	Q2

Figure 4: The Turing machine of Figure 3 in matrix form; the initial state is `Q0`. We propose this as a more readable alternative for the standard set of quintuples.

5. Matrix Code

The main application of DSMs is what we call "Matrix Code". This is a DSM with components (K, D, δ, S, H) where D is defined by declarations in a conventional programming language, say P, and where the binary relations in the transition matrix δ are specified by P with reference to the declarations for D.

To use DSMs to best effect, P should be equal to `C` or `C++` in speed and compactness of compiled code. Although the binary relations constituting δ are defined by pieces of code of P, δ itself is not a construct of P — after all, δ is a *matrix*. Thus DSMs of this kind constitute a different programming language, and it is this programming language that we call *Matrix Code*. A specific δ will be referred to as a *code matrix*.

A code matrix is a hybrid object composed of two programming languages: Matrix Code and a conventional programming language P. The primitive binary relations of the matrix elements are written in P. The way they are composed into composite matrix elements as well as the matrix as a whole are written in Matrix Code.

As we shall show, Matrix Code has two advantages over conventional languages: its programs can be their own proof of partial correctness and it supports the parallel development of correctness proof and code. At the same time, a code matrix can be written in such a way as to be trivially translatable to P. One can say that suitably written code matrices are "almost executable". For the examples in this paper we use the following translation method.

The control state is represented by a variable. Each column is translated to a switch on this variable. Each cell in the column is then translated to one of the cases of the switch. The matrix as a whole is translated to the body of a `void` function. The data state of the code matrix becomes the parameter(s) of the function.

The elements of a code matrix are binary relations over the data states, as in all DSMs. In the case of Matrix Code these binary relations are often composed of primitive relations, which are of two kinds: guards and statements. Guards are boolean expressions; semantically they are subsets of the identity relation. That is, if b is a boolean expression, then its meaning is $\{(d, d) \mid b \text{ is true in data state } d\}$.

The primitive relations that are not guards are statements of P. If S is a statement, then its meaning is the set of pairs (d, d') such that d' is a possible state of termination of S if S starts execution in data state d. Because guards and statements both denote binary relations over D, they are freely intercomposable:

```
    guard ; guard
statement ; guard
statement ; statement
    guard ; statement
```

are all defined, and are binary relations over D. As far as Matrix Code is concerned, `x-- ; x >= 0` and `x > 0 ; x--` are equally valid expressions for binary relations. The latter form is preferred for reasons of translatability to a conventional programming language.

Example: code matrix for computing prime numbers. Consider a DSM with components (K, D, M, S, H) with $K = \{A, B, C, H, S\}$ and D the set of tuples with as components an integer `N`, an array `p` of length `N`, and integers `j`, `k`, and `n`. M is the code matrix shown in Figure 13. For example, $M[C, C]$ is the composition of three binary relations:

```
j%p[n+1] == 0 ; j += 2 ; n = 0
```

where `j%p[n+1] == 0` is a boolean expression; therefore a guard and `j += 2` and `n = 0` are statements.

See Figure 5 for an example of a computation.

```
control | data                         N = 3
state   | state
        | k     j      n     p
-----------------------------------
      S | ?     ?      ?     {?,?,?}
      A | 2     ?      ?     {2,3,?}
      B | 2     5      0     {2,3,?}
      C | 2     5      0     {2,3,?}
      B | 2     5      1     {2,3,?}
      A | 3     5      1     {2,3,5}
      H | 3     5      1     {2,3,5}
```

Figure 5: Example of the computation for `N` equals 3 of the code matrix in Figure 13.

If a row is empty, then its control state can occur only in the first state of any computation. Such a control state is the start state. Thus any row indexed by `S` is empty, and is omitted. Similarly, the necessarily empty column indexed by `H` is omitted. In Figure 13, S is the start state and H is the halt state.

Example: code matrix for FSM. In Figure 2 we presented an FSM for recognizing a simple form of decimal numerals. We may regard the input of an FSM, together with an indication of how far it has been read, as the data state of a DSM where the control states are the states of the FSM. Thus the data state has the form of a file to be read sequentially.

Consider the function `decNum` in Figure 6. It is really two programs in one. If we disregard the line commented with `data state component Aux` and all assignments to the variables declared there, then the remaining part of the data state (`data state component FSM`) is just enough to simulate an FSM: the variable `inp` contains the first symbol of the part of the input that has not been processed. By executing `inp = tape.rd()` the input is advanced by one symbol, so that `inp` is once again the first symbol of the part of the input not processed.

Under the exclusive consideration of `data state component FSM` the function `decNum` decides only whether the input is a legitimate decimal number according to the FSM in Figure 2. Usually more is wanted: one might want to know the value of the decimal numeral. The advantage of writing the FSM in the form of a dual-state machine as in Figure 6 is that one needs to extend only the data state (in this figure to include `data state component Aux`) and to add appropriate operations on the extended data state to ensure that by the time the input is accepted, the value of the decimal numeral read is in the data state component `val`.

The structure of the function `decNum` reflects the matrix in Figure 2. The outer `switch` translates the matrix column by column. Each of the inner `switch` statements translates the contents of the column concerned by code activated by the content of `inp`.

```
int decNum(Tape& t) {
 const int S=0, A=1, B=2, H=3;
 int state = S; // control state
 char inp = t.rd(); // data state component FSM
 int sign, val; // data state component Aux
 while (true) {
  switch (state) {
   case S:
    switch (inp) {
      case'+': sign = +1; inp = t.rd(); break;
      case'-': sign = -1; inp = t.rd(); break;
      default: sign = +1;
    } state = A; break;
   case A: switch (inp){
    case'0':case'1':case'2':case'3':case'4':
    case'5':case'6':case'7':case'8':case'9':
    val = inp - '0';
    inp = t.rd(); break;
    default: assert(false);
    // other character in inp not OK
   } state = B; break;
   case B: switch (inp) {
    case'0':case'1':case'2':case'3':case'4':
    case'5':case'6':case'7':case'8':case'9':
    val = 10*val + (inp-'0');
    inp = t.rd(); state = B; break;
    default: state = H;
   } break;
   case H: return sign*val;
  }
 }
}
```

Figure 6: Translation to `C++` of DSM simulation of FSM of Figure 2. The data state is in the `Tape& t` of the first line. The object that is the content of the formal parameter `t` encapsulates the input tape of the FSM. When the FSM arrives in a state where the first symbol of the input is consumed, then the `C++` statement `inp = t.rd()` advances the input tape and makes `inp` again the first symbol of the remaining input tape.

```
void turing(Tape& t) {
  typedef enum{H,Q0,Q1,Q2} State;
  State state(Q0); // control state
  while (true) {
    switch (state) {
      case Q0:
        if (t.r() == '(') {t.w('('); t.d(R); state = Q0;} else
        if (t.r() == 'X') {t.w('X'); t.d(R); state = Q0;} else
        if (t.r() == ')') {t.w('X'); t.d(L); state = Q1;} else
        if (t.r() == 'A') {t.w('A'); t.d(L); state = Q2;}
        break;
      case Q1:
        if (t.r() == '(') {t.w('X'); t.d(R); state = Q0;} else
        if (t.r() == ')') {t.w(')'); t.d(L); state = Q1;} else
        if (t.r() == 'X') {t.w('X'); t.d(L); state = Q1;} else
        if (t.r() == 'A') {t.w('0'); t.d(d); state = H;}
        break;
      case Q2:
        if (t.r() == 'X') {t.w('X'); t.d(L); state = Q2;} else
        if (t.r() == ')') {/*can't happen*/ assert(false);} else
        if (t.r() == '(') {t.w('0'); t.d(d); state = H;} else
        if (t.r() == 'A') {t.w('1'); t.d(d); state = H;}
        break;
      case H: return;
      default: assert(false); // can't happen
} } }
```

Figure 7: Code version of the Turing machine in Figures 3 and 4. The code is in the form of a C++ function with as single argument t an abstract data to represent the tape. The abstract data type allows three functions: for reading (called as t.r), for writing (called as t.w), and for setting tape direction (called as t.d).

6. Verification of Matrix Code

Verification of Matrix Code is based on Hoare's verification method for conventional code [12], which in turn is based on Floyd's verification method for flowcharts [10]. In this section we review Hoare's method, then show that it can be generalized to binary relations over any domain, which in turn gives a verification method for Matrix Code.

6.1. Hoare's verification method for conventional code

As an introduction to the verification method for imperative programming due to Hoare [12] we verify a Java version of the prime-number generating program developed by Dijkstra in [9]. The Java version of this program is shown in Figure 8.

```
public static void primes(int[] p, int N) {
  // S
  int j,k,n;
  p[0] = 2; p[1] = 3; k = 2;
  // A
  while (k<N) {
    j = p[k-1]+2; n = 0;
    // B
    while (p[n]*p[n] <= j) {
      // C
      if (j%p[n+1] != 0) n++;
      else {j += 2; n = 0;}
    }
    p[k++] = j;
  }
  // H
}
```

Figure 8: A Java function for filling `p[0..N-1]` with the first `N` primes. At the points indicated by the comments S, A, B, C, H we need conditions to allow verification by Hoare's method. The identifiers and the structure are the same as in Dijkstra's example [9].

The concept of configuration consisting of a control state and data state used to define the semantics of DSMs applies to conventional code as well. Here the control state is a code location and the data state is the tuple of values of the variables. According to Hoare's method, conditions are attached to code locations. The conditions make assertions about program variables. When such a condition occurs in a loop, it is the familiar invariant of that loop. In Figure 8 we have indicated by the comments `S`, `A`, `B`, `C`, and `H` where these conditions have to be placed. Figure 9 contains the corresponding conditions.

```
Conditions:
S: p[0..N-1] exists and N>1
H: p[0..N-1] are the first N primes
A: S && p[0..k-1] are the first k primes && k <= N
B: A && k<N && relB(p, k, n, j)
C: B && p[n]*p[n] <= j

relB(p,k,n,j) means that there is no prime
between p[k-1] and j, and that j is not divided
by any prime in p[0..n], and that n<k.

Hoare triples:

{S} p[0]=2; p[1]=3; k=2; {A}
{A && k >= N}  {H}
{A && k < N} j=p[k-1]+2; n=0; {B}
{B && p[n]*p[n] <= j} {C}
{B && p[n]*p[n] > j} p[k++] = j {A}
{C && j%p[n+1] != 0} n++ {B}
{C && j%p[n+1] == 0} j += 2; n = 0 {B}
```

Figure 9: Conditions and Hoare triples for Figure 8. The meaning of a Hoare triple {`A0`} `CODE` {`A1`} is that if condition `A0` is true and if `CODE` is executed with termination, then condition `A1` is true.

The verification of the function as a whole relies on the verification of a number of implications defined in terms of conditions and program elements such as tests and statements. Consider Figure 8: because there is an execution path from `A` to `B`, one has to show the truth of

```
{A && k<N} j=p[k-1]+2; n=0; {B},
```

which has as meaning: if `A && k<N` (the *precondition*) is true and if

```
j=p[k-1]+2; n=0;
```

is executed, then `B` (the *postcondition*) is true. Because of the three elements: precondition, postcondition, and the item in between, this is called a *Hoare triple*. Figure 9 contains not only the conditions for Figure 8, but also the set of verification conditions in the form of Hoare triples.

The term "condition" for the type of thing that occurs as precondition and postcondition in a Hoare triple is, in our view, rather compelling. However, it seems that in certain contexts "assertion" is a more natural alternative term. In this paper we will use both. At the same time, one should make a distinction between the condition as a linguistic expression and the set that is the meaning of that expression. We trust no confusion arises as we use “assertion” and “condition” interchangeably for both the expression and the meaning.

According to Hoare’s method the program in Figure 8 is verified by the truth of the Hoare triples in Figure 9. Why the set of conditions used is necessary and sufficient and how partial correctness follows from the truth of the Hoare triples requires a non-negligible amount of explanation. This can be omitted here because in the following we give the equivalent explanation for Matrix Code, for which it will be comparatively simple.

6.2. Binary relations, conditions, and Hoare triples

Let us consider a binary relation R, a subset of $D \times D$, where we can think of D as a set of data states. Let us call subsets of D *conditions*. The *left projection* of R is defined as the condition $\{x \in D \mid \exists y \in D.\ (x, y) \in R\}$. Dually, the *right projection* of a binary relation R is defined as the condition $\{y \in D \mid \exists x \in D.\ (x, y) \in R\}$.

We generalize I_D to I_c, which means, for any condition $c \subseteq D$, by definition, $\{(x, x) \in D \times D \mid x \in c\}$. This induces a one-to-one relation between c and I_c:

$$x \in c \leftrightarrow (x, x) \in I_c.$$

Accordingly, at times we view a condition (alias *assertion*) as a subset of D; at times as a subset of I_D.

Definition 6. *Given a condition $c \subseteq D$ and a binary relation $R \subseteq (D \times D)$, we write* $\{\mathtt{c}\}\mathtt{R}$ *for the right projection of $I_c; R$, where I_c is the binary relation $\{(x, x) \in D \times D \mid x \in c\}$.*

As we saw above, Hoare triples were intended to be applied to program statements. Here we see that they have a natural interpretation for binary relations.

Definition 7. *Given conditions $p \subseteq D$ and $q \subseteq D$ and a binary relation $R \subseteq (D \times D)$, we define that* $\{\mathtt{p}\}\mathtt{R}\{\mathtt{q}\}$ *(*the Hoare triple*) holds iff*

$$\{\mathtt{p}\}\mathtt{R} \subseteq \mathtt{q}.$$

That is, if the input to R satisfies $\mathtt{p}$, then all corresponding outputs (if any) satisfy $\mathtt{q}$.

We extend Definition 6 to vectors and matrices.

Definition 8. *Let v be a vector of conditions: $v \in K \to 2^D$ and let M be a matrix of binary relations: $M \in K \times K \to 2^{D \times D}$. Then $\{v\}M$ is defined to be the vector in $K \to 2^D$ such that $(\{v\}M)[i] = \bigcup_{j \in K} \{v[j]\}M[j,i]$.*

Definition 9. *Let $p, q \in K \to 2^D$ be vectors of conditions indexed by K and $M \in K \times K \to 2^{D \times D}$ a matching matrix of binary relations over D. The expression $\{p\}M\{q\}$ asserts that $(\{p\}M) \subseteq q$ where the set inclusion is taken elementwise.*

Theorem 3. *Given a code matrix M and a condition vector V satisfying $\{V\}M\{V\}$. For any configuration (k', d') of any computation beginning with (k, d) such that $d \in V[k]$ it is the case that $d' \in V[k']$.*

Proof
We proceed by induction on the length n of the computation. If $n = 1$ (one transition in the computation) we have $(k', d') = (k, d)$. Assume the theorem true for computations of length $n - 1$. Consider the computation

$$(k, d), (k_1, d_1), \ldots, (k_{n-1}, d_{n-1}), (k', d').$$

By the induction assumption $d_{n-1} \in V[k_{n-1}]$. We have that $(d_{n-1}, d') \in M[k_{n-1}, k']$. It is given that `{V}M{V}`, hence in particular that

$$\{V[k_{n-1}]\}M[k_{n-1}, k']\{V[k']\}.$$

It follows that $d' \in V[k']$, which establishes the theorem for the computation of length n.

7. Parallel development of proof and code

Floyd's method is difficult to apply because it is difficult to find the required conditions. Because of this Dijkstra [7, 8] advocated parallel development of code and proof. In this section we demonstrate parallel development of a code matrix for the sample problem solved in Figure 8: to fill an array with the first N prime numbers in increasing order.

Background on prime numbers. Before we start, let us review what we need to know about prime numbers. The following list of facts is not intended as a complete or nonredundant set of axioms; they are a selection to guide us in the choice of conditions and transitions.

1. *A prime is a positive integer that has no divisors.* (We do not count 1 or the integer itself as divisors. Moreover, 1 is not a prime.)

2. *There are infinitely many primes*, so the problem can be solved for any N.
3. *2 and 3 are the first two primes.* So a way to get started is to accept these as given and place them in the beginning of the table. This has the advantage that we always have the situation where the last prime in the table is odd and the next odd number is the first candidate to be tested for the next prime.
4. *If a number has a divisor, then it has a prime divisor.* This can be used to save effort: we have to test only for divisibility by smaller primes, and these are already in the table.
5. *If a number has a divisor, then it has a prime divisor less than or equal to its square root.* This implies that we do not have to test the candidate for the next prime for divisibility by all primes already in the table.
6. *The square of every prime is greater than the next prime.* The significance of this fact will become apparent as we proceed.

Deriving the code matrix. The distinctive advantage of Matrix Code is that a matrix can be expanded from the specification in small steps using only the *logic* of the application without needing to attend to the *control* component of the algorithm. Thus Matrix Code is an example of Kowalski's principle "Algorithm = Logic + Control" [13].

We assume that the specification exists in the form of a precondition and a postcondition. This gives rise to code matrix with one row and one column; the one in Figure 10.

`S: p[0..N-1] exists & N>1`	
`/*which T?*/`	`H: p[0..N-1] contains the first N primes`

Figure 10: There is only an empty transition T such that {`S`}`T`{`H`}.

The one element of this matrix is the transition `T` such that {`S`}`T`{`H`} is true. That is, `T` has to be a simple combination of guards and assignment statements that places the `N` first primes in `p`, whatever `N` is. Absent such a `T`, we leave the matrix cell empty. The resulting code matrix satisfies {`S`}`T`{`H`}, which makes it partially correct, but *very* partially so: it has no successful computations. Although Figure 10 is the correct start of the development process, it is not the last step.

As it is too ambitious to place all primes in the array with a single transition, a reasonable thing to try is to fill it with the first `k` primes and then try to add the next prime after `p[k-1]`.

We need a condition `A` that is intermediate in the sense that {`S`}`T1`{`A`} and {`A`}`T2`{`H`} for simple `T1` and `T2`. Such a condition is: the first `k` primes in increasing order are in `p[0..k-1]` with `1 < k <= N`.

Condition `A` is promising because it is easy to think of such a `T1` and such a `T2`. The result is in Figure 11.

`A:`	`S: p[0..N-1] exists & N>1`	
`k >= N`		`H: p[0..N-1] contains the first N primes`
	`p[0] = 2; p[1] = 3; k = 2`	`A: p[0..k-1] contains the first k primes & k <= N`

Figure 11: In column *A* the case `k < N` is missing.

This again is a partially correct code matrix. It is a slight improvement in that it solves the problem if N happens to be one or two. In all other cases it leads to failed computations. The difficulty is that in column `A` we may have that `k < N`, so that we cannot make the transition to `H`. We need to find the next prime after `p[k-1]`. Let `j` be the current candidate for this next prime. That suggests for condition `B:` `A` is true and `j` is such that there is no prime greater than `p[k-1]` and less than `j`.

This is always true when `j` is the next odd number after `p[k-1]`. Another way of saying this is that `j` is not divisible by any of the primes in `p[0..n]` with `n` set to 0. We are interested more generally in

> There are no primes between `p[k-1]` and `j` (with `j` is not divisible by any of the primes in `p[0..n]`) and `n<k`.

We abbreviate this condition to `relB(p,k,n,j)`.

The largest prime factor of a number is less than the square root of the number. Hence, if we find that the square of `p[n+1]` is greater than `j`, then we can conclude that `j` is the next prime after `p[k-1]`. Hence, in the new column `B`, it is easy to detect whether `n` is large enough to conclude that `j` is the next prime after `p[k-1]`. We place the corresponding transition in column `B` and we have Figure 12.

B:	A:	S: p[0..N-1] exists & N>1	
	k >= N		H: p[0..N-1] contains the first N primes
p[n]*p[n]>j; p[k++]=j		p[0] = 2; p[1] = 3; k = 2	A: p[0..k-1] contains the first k primes & k <= N
	k<N; j = p[k-1]+2; n=0		B: A & k<N & relB(p,k,n,j)

Figure 12: In column A we have added a transition in column A for the case that `k < N`. In that case we can start finding the next prime after `p[k-1]` because we know that there is enough space in `p` to store it. `relB(p,k,n,j)` means that there is no prime between the last prime found and `j` and that `n<k`, and that `j` is not divided by any prime in `p[0..n]`.

There are still failed computations. (In fact, there is still no way to get beyond $N = 2$.) The way ahead is clear: a transition is missing in column `B`, for the situation where `n` is too small to conclude that `j` is the next prime. That in itself produces condition `C` and, with it, a new row and column.

In column `C` the missing information is whether `j`, the candidate for the next prime, is divisible by `p[n+1]`. If not, then `n` can be incremented, and condition B is verified. If so, then j is not a prime and the search for the next prime must be restarted with j+2. This determines a transition in column C that verifies condition C, so is placed in that row. See Figure 13.

C:	B:	A:	S: p[0..N-1] exists & N>1	
		k >= N		H: p[0..N-1] contains the first N primes
	p[n]*p[n]>j; p[k++]=j		p[0] = 2; p[1] = 3; k = 2	A: p[0..k-1] contains the first k primes & k <= N
j%p[n+1]!=0; n++		k<N; j = p[k-1]+2; n=0		B: A & k<N & relB(p,k,n,j)
j%p[n+1]==0; j += 2; n=0	p[n]*p[n]<= j			C: B & p[n]*p[n] <= j

Figure 13: This figure is both a general example of a code matrix and the final stage of the development consisting of the sequence of Figures 10, 11, and 12. Change from Figure 12: row and column with label *C* are added. There are no incomplete columns. This, as well as each of the previous versions is partially correct, as implied by the validity of the verification condition for each of the null matrix elements. The absence of incomplete columns opens the possibility of total correctness, but does not prove it.

Up till now we detected with every additional row and column that the new column lacked a transition. Not this time: none of the columns has a missing transition. The code matrix has no failed computations. So it gives the correct answer by exiting in row H, or it continues in an infinite computation. As we have proved only partial correctness, this latter alternative remains a possibility.

Termination. For an infinite computation to arise, there must be at least one condition that is revisited an infinite number of times. For each condition we give a reason why it can be revisited only a finite number of times.

1. Condition `A.` For this condition to be returned to, `k` has to have increased. `k` is never decreased and is bounded by `N`.
2. Condition `B.` For this condition to be returned to, `n` or `j` has to have increased. `n` is bounded by the square root of `p[N-1]`. The number of times it is reset to zero is bounded by `p[N-1]`. `j` is never decreased and is bounded by `p[N-1]`.
3. Condition `C.` For this condition to be returned to, `n` has to have increased and is bounded as noted above.

The transitions have been chosen so that the corresponding revisiting condition is satisfied. As none of these conditions can be satisfied an infinite number of times, the code matrix has no infinite computation.

Running Matrix Code. Running a code matrix in current practice requires translation to a currently available language. Our examples of Matrix Code have been constructed for ease of translation to languages like Java or `C`. This entails a drastic reduction in expressivity. Let us now demonstrate translation using Figure 13 as example.

As there is a similarity between the control states and the states of a finite-state machine (FSM), a good starting point for systematic translation of a code matrix is the pattern according to which an FSM is implemented. This is usually done by introducing a constant for every state and to let a variable, say, `state` assume these constants as values. An infinite loop containing a `switch` controlled by `state` then contains a `case` statement for every control state.

Each column of a code matrix translates to a `case` statement. The order in which the translations of the columns occur does not matter as long as `state` is initialized at `S`. Here we have arbitrarily chosen alphabetic order. In this way Figure 13 translates to the following.

A transition `b0;S0` in column X and row R_0 and transition `!b0;S1` in column X and row R_1 translate to `case X: if (b0) {S0; state = R0;} else {S1; state = R1} break;` in the above code.

8. Expressiveness of Matrix Code

The code obtained by translating a code matrix is quite different from what one conventionally would write: compare Figure 8 with Figure 14. In

```
void prTable(int p[], int N) {
  typedef enum{A,B,C,H,S} State;
  State state(S); // control state
  int j,k,n;      // part of data state
  while (true) {
    switch(state) {
      case A:
        if(k >= N) state = H;
        else {j = p[k-1]+2; n = 0; state = B;}
        break;
      case B: if (p[n]*p[n] > j) {
                p[k++] = j; state = A;
              } else state = C;
              break;
      case C:
        if (j%p[n+1] != 0) {n++; state = B;}
        else {j += 2; n = 0; state = C;}
        break;
      case H: return;
      case S: p[0] = 2; p[1] = 3; k = 2; state = A;
    }
  }
}
```

Figure 14: Translation of the code matrix in Figure 13 to C++.

this example Matrix Code has the advantage of being a verification and of being easy to discover. But in the prime-number problem Matrix Code does not lead to a more efficient program: it has the same set of computations as the conventional one.

In this section we present an example where Matrix Code makes it easy to discover an algorithm that is more efficient than what is obtained via the conventional programming style. Consider the merging of two monotonically nondecreasing input streams into a single output stream. We have available the following `C++` functions.

```
bool getL(int& x);  // output parameter x
bool getR(int& x);  // output parameter x
void putL();
void putR();
```

where `getL` (`getR`) tests the left (right) input stream for emptiness. In case of nonemptiness the output parameter `x` gets the value of the first element of the stream. Neither `getL` nor `getR` change any of the streams. This is done only by the functions `putL()` and `putR()` which transfer the first element of a nonempty left or right input stream to the output stream.

Figure 15 is a typical program for this situation. It typically acts in two stages. In the first stage both input streams are nonempty. In the second stage one of the input streams is empty so that all that remains to be done is to copy the other stream to the output.

```
void eMerge() {
  int u,v;
  while (getL(u) && getR(v))
    if (u <= v) putL();
    else        putR();
  while (getL(u)) putL();
  while (getR(v)) putR();
}
```

Figure 15: A structured program for merging two streams.

This algorithm performs unnecessary tests: in the first stage only one of the input streams is changed, so that only that one needs to be tested for emptiness; here both are tested[2]. It is superfluous tests like this that allow the algorithm to be as simple as it is.

Of course it is unlikely that it is important to save the kind of test just mentioned. But there are many types of merging situations and there may be some in which it does matter. An advantage of Matrix Code is that it does not bias the programmer towards including superfluous tests.

We proceed to develop a code matrix for merging. The assertions need to indicate whether it is known that an input stream is empty and, if not, what its first element is. If an input stream is possibly empty then we represent it by "`?`". We write "`e`" if an input stream is empty. Nonemptiness is indicated by writing "`x:?`", where `x` is the first element. We have to do this for each of the input streams; we write e.g. the assertion `(u:?,v:?)` to mean that both input streams are nonempty and have first elements `u` and `v`, respectively.

We write all conditions in the form `(left,right)`, where `left` and `right` indicate the state of the input concerned, in conjunction with the statement that the result of appending the output to the result of merging the remaining input streams is equal to the result of merging the input streams before the beginning of the execution of the program. As this conjunct is part of every condition, it need not be stated explicitly. Of course its validity needs to be verified for every matrix entry.

With these conventions we can state the program's specification as obtaining a transition from the state `S`, which is `(?,?)` to the state `H`, which is `(e,e)`. Accordingly, the development starts with Figure 16.

`S:(?,?)`	
`/*which T?*/`	`H:(e,e)`

Figure 16: Matrix Code corresponding to specification of the merging program. But there is no `T` such that {`S`}`T`{`H`}. The conditions in this figure, as well as those in Figures 17 and 18 include the unstated conjunct that the result of appending the output stream to the merge of the input streams is equal to the merge of the input streams in the initial state.

As always with Matrix Code, we start with the conditions. Which do we need, in addition to the `(?,?)` and `(e,e)` given by the specification? For

[2]With the one exception when the left input stream runs out at the same time as, or before, the right input stream.

each of the input streams there are three states of information:

- `?`
- `e`
- `x:?` for some first element `x`

It is to be expected that the two input streams can assume each of the three information states independently, for a total of nine conditions.

It is desirable that the initial condition `(?,?)` of minimal information does not arise during a computation of the code matrix. Under the assumption that we can avoid this there will be only rows for the eight other conditions. By the time we will have populated the columns for these eight conditions we will see whether this assumption was justified.

This problem is easy because the conditions are determined by the nature of the problem. For each condition there is an obvious and easy-to-realize revisiting condition. If there is at least one unknown input stream at least one of them has to become known before revisiting. If both input streams are known, then at least one of them has to have its first element transferred to output before revisiting. See Figures 17 and 18, where the transitions have been chosen to conform to the revisiting requirements. As each column either has no guard or two complementary guards, no additional rows are needed.

The translation of the code matrix in Figure 18 to `C++` is given in Figure 19. As the order of the translations of the columns is immaterial, we have placed them in alphabetic order by label.

The reason for developing a code matrix for the merge problem was the desire to avoid the superfluous tests of a function like the `eMerge` listed in Figure 15. To see in how far `mMerge` improves in this respect we have run both functions on the same set of pairs of input streams and counted the calls executed in both merge functions.

Such comparisons are of course dependent on the nature of the input streams. For example, the more equal in length the input streams are, the more favourable for `mMerge`. Accordingly we have used a random-number generator to determine the lengths of the input streams. The input streams themselves are monotonically increasing with random increments.

`A`	`S:(?,?)`	
		`H:(e,e)`
	`getL(u)`	`A:(u:?,?)`
	`!getL(u)`	`B:(e,?)`
`getR(v)`		`C:(u:?,v:?)`
`!getR(v)`		`D:(u:?,e)`

Figure 17: See Figure 16. An input stream needs to be tested; the left one is chosen arbitrarily. This gives rise to new conditions. Columns for these will cause addition of yet more conditions. See Figure 18.

	getL	getR	putL	putR
eMerge	1756	2691	871	1819
mMerge	872	1821	871	1819
eMerge	1067	830	655	410
mMerge	656	411	655	410
eMerge	3261	735	2894	365
mMerge	2895	366	2894	365
eMerge	1355	1024	844	509
mMerge	845	510	844	509

Each pair of successive lines gives the result of running `eMerge` and `mMerge` on the same pair of input streams. The lengths of the streams are not listed separately, as they are equal to the number of calls to `putL` and `putR` shown in the table.

A merge function needs to make at least one call to `getL` (`getR`) for every element of the left (right) input stream. It can be seen that `mMerge` remains close to this minimum, while `eMerge` does not.

This example is notable in that Matrix Code yields an unfamiliar, test-optimal algorithm by *default*. Structured programming tends to reduce the number of control states. Matrix Code lacks this bias: in its use it is natural to introduce control states as needed to serve as memory for test outcomes.

G	F	E	D	C	B	A	S:(?,?)	
	!getL(u)				!getR(v)			H:(e,e)
				u>v; putR()			getL(u)	A:(u:?,?)
putR()					getR(v); putR()		!getL(u)	B:(e,?)
		getL(u)				getR(v)		C:(u:?,v:?)
	getL(u)					!getR(v)		D:(u:?,e)
				u <= v; putL()				E:(?,v:?)
			putL()					F:(?,e)
		!getL(u)						G:(e,v:?)

Figure 18: The complete code matrix for the merging problem, continuing Figures 16 and 17.

```
void mMerge(Trinity& tri) {
  int u,v;
  typedef enum{S,A,B,C,D,E,F,G,H} State;
  State state = S; // control state
  while(true) {
    switch(state) {
      case A: state = (tri.getR(v))?C:D; break;
      case B: if (tri.getR(v)) {tri.putR(); state = B;}
         else state = H; break;
      case C: if (u <= v) {tri.putL(); state = E;}
         else {tri.putR(); state = A;} break;
      case D: tri.putL(); state = F; break;
      case E: state = tri.getL(u)?C:G; break;
      case F: state = tri.getL(u)?D:H; break;
      case G: tri.putR(); state = B; break;
      case H: return;
      case S: state = tri.getL(u)?A:B; break;
    }
  }
}
```

Figure 19: A `C++` function for merging two streams translated from Figure 18. `tri` is an object of class `Trinity`. It contains three components: two input streams and an output stream. The admissible operations on these components are `getL(u)`, which is to determine the value of the first element from the left input stream (if there is one) and to make the argument `u` equal to it. The input stream is left unchanged. `putL()` removes the first element from the left input stream and makes it the next element of the output stream. Similarly for `getR(v)` and `putR()` for the right input stream.

9. Related work

We organize related work in the form of seven ways to discover Matrix Code: flowcharts, automata theory, abstract state machines, augmented transition networks, logic programming, tail-recursion optimization, and recursive program schemes.

Flowcharts. The following comment has been made on Matrix Code: "*Although it reeks of flowcharts, the proposal has some merit.*" The comment has some merit: flowcharts are indeed closely related to Matrix Code. Flowcharts were widely used as an informal programming notation from the early 1950s to 1970. Floyd [10] showed how assertions and verification conditions can prove a flowchart partially correct. Hoare [12] introduced the notation of triples for the verification conditions and cast Floyd's method in the form of inference rules for control structures such as

`while ... do ...` and `if ... then ... else ...`

Dijkstra observed that verifying assertions are difficult to find for existing code, so that an attempt at verification is a costly undertaking with an uncertain outcome. He argued [7, 8] that code and correctness should be "developed in parallel". The proposal seems to have found no response, if only for the lack of specifics in the proposal.

Given the fact that Dijkstra's proposal was considered unrealistically utopian, and still is, it is interesting to read what seems to be the first treatise [11] on programming in the modern sense, published in 1946. Here programs are expressed in the form of *flow diagrams.* At first sight one might think that these are flowcharts under another name. This is not the case: flow diagrams consist of executable code integrated with assertions, with the understanding that a consistent flow diagram proves the correctness of the computations performed by it.

The imperative part of a flow diagram was translated to machine code (this was before the appearance of assemblers). I found no indication in [11] that it was even contemplated to split off the imperative part of the flow diagram. Thus we see that what was a vague proposal [7, 8], and regarded as unrealistically utopian in 1970, was fully worked out in 1946 and may have become a practical reality in 1951 when the IAS machine became operational.

By the time flowcharts appeared, the proof part of flow diagrams had been dropped. And apparently forgotten, for Floyd's discovery was published in 1967 and universally acknowledged as such. Floyd's format is rather different, and, in our opinion, preferable to the flow diagrams of [11]. Matrix

Code can be regarded as a simplification of Floyd's flowchart annotated with assertions, a simplification made possible by the use of binary relations that provide a common generalization of statements and tests. Apt and Schaerf unify statements and tests in their nondeterministic control structures [1].

Automata theory. DSMs can be regarded as a realization of Dana Scott's idea [17] to put an end to the proliferation of new variations of FSM by replacing them by programs defined to run on suitably defined computers. DSMs are very different from the programs proposed by Scott. Scott's programs are unlike FSMs; DSMs closely resemble FSMs. Paradoxically, DSMs, in the form of Matrix Code, are of practical use; Scott's programs are not.

Abstract State Machines. DSMs can be obtained as a drastic simplification of ASMs [3] where evolving algebras are replaced by binary relations over data states and formulas of logic are replaced by guards. One might think that guards are a special case of the formulas of the ASMs. There is however a fundamental difference: regarded as logic formulas, guards have free variables; the formulas of ASMs do not.

Augmented Transition Networks. In spite of Scott's plea [17], variants of FSM continued to appear. Of special interest in this context are *labeled transition systems* which are used to model and verify reactive systems [2]. Here the set of states is often infinite and there is typically no halt state. Such systems are specified by rules of the form $P \xrightarrow{A} Q$ to indicate the possibility of a transition from state P to state Q accompanied by action A. Mathematically the rules are viewed as a ternary relation containing triples consisting of P, A, and Q. This is of course unobjectionable, but the alternative view of the rules as constituting a matrix indexed by states, containing in this instance A as element indexed by P and Q has the advantage of connecting the theory to that of semilinear programming in the sense of Parker [15]. Another variant of FSM are the *augmented transition networks* used in linguistics [19]. The modification of flowcharts by means of binary relations was introduced in [18]. These can be viewed as augmented transition networks with binary relations as labels on the transition arrows.

Logic Programming. The property that a code matrix is both a set of logical formulas and an executable program is reminiscent of logic programming, especially its aspect of separating logic from control [13]. A special form of logic program corresponding to imperative programs was investigated in [4].

Recursive program schemes. De Bakker and de Roever [6] modeled programming constructs such as if-then-else and while-do. For both guards and assignments they used binary relations among what we call data states.

Tail-recursion optimization. An attractive way of deriving efficient imperative code is to use a recursive definition of the function to be computed as starting point. These can sometimes be transformed to a form in which there is a single recursive call and where this call occurs as the last statement of the function. A further transformation replaces this call by the more efficient `goto` statement. The result is similar to the result of translating a code matrix to executable code. The definition of the function can then be used to obtain an assertion verifying the transformed program. This is used in logic programming [4].

10. Conclusions

In this paper we write programs as matrices with binary relations as elements. These matrices can be regarded as transformations in a generalized vector space, where vectors have assertions about data states as elements. Computations of the programs are characterized by powers of the matrix and verified assertions show up as generalized eigenvectors of the matrix. Such results may be dismissed as frivolous theorizing. It seems to us that they are related to the following practical benefits.

Our motivation was to address the fact that imperative programming is in an unsatisfactory state compared to functional and logic programming. In the latter paradigms, implementation is, or is close to, specification. In imperative programming the relation between implementation and specification is the verification problem, a problem considered too hard for the practising programmer. We proposed Matrix Code as an imperative programming language where the same construct can be read as logical formula and can serve as basis for a routine translation to Java, `C`, or `C++`.

Matrix Code is only applicable to small algorithms. Take it as a warning sign when it no longer fits on the back of an envelope. Yet it can play a useful role in large programs. Even the largest software system is ultimately subdivided into functions or methods. Software engineering wisdom is unanimous in declaring any function that is not small as a "code smell" and hence a candidate for refactoring. Everyone of these many small functions is a candidate for derivation by Matrix Code.

Experience so far suggests that it is possible to develop algorithms incrementally by small, obvious steps from the specification. In this paper we go through such steps for an algorithm to fill a table with prime numbers using the method of trial division. Whether or not this success is an exceptional case, it seems certain that progress has been made in the direction of the old dream according to which the production of verified code is facilitated by developing proof and code in parallel.

Acknowledgments

Thanks to Paul McJones and Mantis Cheng for their help. I am grateful to the reviewers for PPDP 2012 and to the ones for *Science of Computer Programming* for their careful reading and for their suggestions for improvement. This research benefited from facilities provided by the University of Victoria and by the Natural Science and Engineering Research Council of Canada.

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